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Parametrizing open universals

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Abstract

All spaces are assumed to be regular Hausdorff topological spaces. If X and Y are spaces, then an open set U in $X \times Y$ is an open universal set parametrized by Y if for each open set V of X, there is $y \in Y$ such that $V = \{x \in X\colon (x, y) \in U\}$. A space Y is said to parametrize $\mathcal{W}(\kappa)$ if Y parametrizes an open universal set of each space of weight less than or equal to κ. The following are the important results of this paper.

If a metrizable space of weight κ parametrizes $\mathcal{W}(\kappa)$, then κ has countable cofinality. If κ is a strong limit of countable cofinality, then there is a metrizable space of weight κ parametrizing $\mathcal{W}(\kappa)$. It is consistent and independent that there is a cardinal κ of countable cofinality, but not a strong limit, and a metrizable space of weight κ parametrizing $\mathcal{W}(\kappa)$.

It is consistent and independent that a zero-dimensional, compact first countable space parametrizing itself (equivalently, parametrizing all spaces of the same or smaller weight) must be metrizable.

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1. Introduction

Let X and Y be spaces. Then Y is said to *parametrize* X if there is an open subset U of $X \times Y$ (called an *open universal*) such that every open subset of X occurs as a horizontal slice, $U^y = \{x \in X\colon (x, y) \in U\}$, for some y in Y. More generally, if $\mathcal{C}$ is a class of topological spaces, then Y parametrizes $\mathcal{C}$, if every X in $\mathcal{C}$ is parametrized by Y.

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In [5,4] we investigated how the topological properties of X are affected by having an open (or, more generally, Borel) universal parametrized by a 'nice' space Y. In the present paper, the attention shifts to the parametrizing space Y, and we seek to determine which spaces will parametrize open universals for all members of some class of spaces. As proved in [5], 2^κ, a compact space of weight and character κ, and $D(2^\kappa)$, a metrizable space (hence first countable and with a G_δ-diagonal) of weight 2^κ, parametrize all members of $\mathcal{W}(\kappa)$, the class of all spaces with weight no more than κ.

The advantage of the Cantor cube, 2^κ, as a parametrizing space, is that it is *weight efficient*. In other words, it has the smallest possible weight to parametrize $\mathcal{W}(\kappa)$. Consider which other spaces of weight κ parametrize open universals for all spaces of weight κ. Any space containing the Cantor cube will do so, and since the parametrization is so natural, the authors originally conjectured that this was the solution: a space of weight κ parametrizes $\mathcal{W}(\kappa)$ if and only if it contains a homeomorph of 2^κ. But we show (Theorem 29) that any κ-Cantor Bernstein set in 2^κ parametrizes $\mathcal{W}(\kappa)$ but does not contain a copy of 2^κ. Consistently, there is a subspace of the Cantor set, 2^ω, of size $\mathfrak{c}$, which can not parametrize an open universal of *any* infinite space (Example 30).

The discrete space of size 2^κ is not weight efficient, but it is metrizable, and hence efficient with respect to character. The aim of this paper is to investigate when spaces can be parametrized in such a way as to combine the good points of the Cantor cube and discrete space parametrizations.

- For which cardinals κ is there a metrizable space of weight κ parametrizing all spaces of weight κ?
- For which cardinals κ is there a first countable, compact space of weight κ parametrizing $\mathcal{W}(\kappa)$?

Summarising the results of Sections 4 and 5, our answers to these two questions are as follows. For any cardinal κ of countable cofinality, say $\kappa = \lim_{n\in\omega}\kappa_n$, define C_κ to be $\prod_{n\in\omega} D(\kappa_n)$. Note that C_κ is a metrizable space, of weight κ, whose definition is independent of the choice of κ_n.

Theorem A. *Let κ be a cardinal.*

(1) *If $cf(\kappa) = \omega$ and κ is a strong limit, then C_κ parametrizes $\mathcal{W}(\kappa)$.*
(2) *If a first countable space, with a G_δ-diagonal, and weight κ, parametrizes $\mathcal{W}(\kappa)$, then κ has countable cofinality.*
(3) (GCH) *The following are equivalent*:
 (a) *There is a first countable space with a G_δ-diagonal and weight κ parametrizing $\mathcal{W}(\kappa)$.*
 (b) *The cofinality of κ is countable.*
 (c) *C_κ parametrizes all spaces of weight no more than κ.*
(4) *It is consistent that*:
 No first countable space of weight $\aleph_\omega$ parametrizes $\mathcal{W}(\aleph_\omega)$ (or even $D(\aleph_\omega)$).
(5) *It is consistent that*:
 $C_{\aleph_\omega}$ parametrizes $\mathcal{W}(\aleph_\omega)$, but $\aleph_\omega$ is not a strong limit.

Part (1) of Theorem B below is immediate from the fact that every uncountable compact metrizable space contains a homeomorph of the Cantor set. Note also the restriction to zero-dimensional spaces in part (2). The authors do not know whether this can be removed.

Theorem B.
(1) *A compact, metrizable space parametrizes $\mathcal{W}(\aleph_0)$ if and only if it is uncountable.*
(2) *It is consistent that*: *Every zero-dimensional, first countable, compact space parametrizing itself is metrizable.*
(3) *It is consistent that*: *There is a zero-dimensional, first countable, compact space of weight $\aleph_\omega$ parametrizing $\mathcal{W}(\aleph_\omega)$.*

In Section 3 we introduce a new family of cardinal invariants: for each cardinal μ, the *μ-weight*, denoted w_μ, of a space X, is the size of a smallest family of open subsets of X, so that every open subset of X is the union of no more than μ members of the family. The connection with the above questions is that a space X has an open universal parametrized by a space Y of weight no more than λ and character no more than μ if and only if $w_\mu(Y) \leqslant \lambda$ (Proposition 10).

Key to all our results is an examination of which spaces can parametrize discrete spaces. By $\mathcal{P}(\kappa)$, we mean the set of all subsets of the discrete space $D(\kappa)$. If a first countable space Y of weight λ parametrizes $D(\kappa)$ then there is a $\mathcal{C} \subseteq \mathcal{P}(\kappa)$ of size no more than λ such that every subset of κ is a countable union of elements of $\mathcal{C}$ [5, Theorem 10]. Conversely, if for a pair of cardinals κ and λ there is such a family, then the metric space $D(\lambda)^\omega$ of weight λ parametrizes $D(\kappa)$ [5, Lemma 2]. This leaves the interesting combinatorial problem (taking $\kappa = \omega_1$ for concreteness), of the minimal λ needed. Clearly $\lambda = 2^{\aleph_1}$ suffices. We can also show that λ must be strictly larger than κ $(= \aleph_1$ here), and under **CH** or **MA** $+ \neg$**CH**, no $\lambda < 2^{\aleph_1}$ works (Theorem 13). But it is consistent to have $\mathcal{C} \subseteq \mathcal{P}(\aleph_1)$ countably generating $\mathcal{P}(\aleph_1)$ (as above) and $|\mathcal{C}| < 2^{\aleph_0}$. In fact a much stronger claim is consistent (Theorem 13(3)). These results have implications for the hereditary cellularity of spaces parametrized by spaces with small character (Theorem 14).

Other results in Section 4 include some showing that the spaces C_κ play a similar role among metrizable spaces as Cantor cubes do in compact spaces. So, for example, any κ-metric Bernstein set in C_κ parametrizes $\mathcal{W}(\kappa)$ although it contains no copy of C_κ (Theorem 24). We also consider which spaces parametrize themselves. Evidently, if a space Z parametrizes $\mathcal{W}(w(Z))$, then Z has a self parametrizing universal. Hence 2^κ, the κ-Cantor Bernstein sets, C_κ and κ-metric Bernstein sets, are all self parametrizing. As recorded above, a first countable space Z with a G_δ-diagonal parametrizing itself must have weight of countable cofinality. We can also show if Z has a G_δ-diagonal and is self parametrizing, then $w(Z) \leqslant hc(Z)$ (Proposition 25). In particular, if Z is hereditarily c.c.c. then it is metrizable. In contrast to the situation with (zero-dimensional) compact spaces (Theorem B(2)), there is, in **ZFC**, a first countable, non-metrizable space Z, with G_δ-diagonal, parametrizing $\mathcal{W}(w(Z))$ (Example 28).

We conclude with a list of open problems, and consider possible future developments.

2. Definitions and background material

Our notation follows that of [3,9]. All spaces are assumed to be regular and Hausdorff. Below are the definitions of the cardinal invariants hereditary cellularity hc and $\widehat{hc}$, hereditary Lindelöf degree hL and $\widehat{hL}$, hereditary density degree hd and $\widehat{hd}$, and weight w, which play a key role in this paper. Then follows a brief discussion of the so-called 'sup = max' problem. Finally we have collected together those results concerning open universal sets from [5], relevant to our present discussion.

Cardinal invariants. Let X be a space. The *weight*, $w(X)$, of X is the minimal size of a base for X. The *hereditary density*, $hd(X)$, of X, is the supremum of all cardinals κ, such that there are different $x_\alpha \in X$ and open neighbourhoods V_α of x_α $(\alpha \in \kappa)$ with the property that $x_\beta \in V_\alpha$ implies $\beta \geqslant \alpha$; while $\widehat{hd}(X)$ is the minimum of all κ for which there are no such left separated sequences of length κ. The *hereditary Lindelöf degree*, $hL(X)$, of X, is the supremum of all cardinals κ, such that there are different $x_\alpha \in X$ and open neighbourhoods V_α of x_α $(\alpha \in \kappa)$ with the property that $x_\beta \in V_\alpha$ implies $\beta \leqslant \alpha$; while $\widehat{hL}(X)$ is the minimum of all κ for which there are no such right separated sequences of length κ. The *hereditary cellularity*, $hc(X)$ of X, is the supremum of all cardinals κ, such that there are different $x_\alpha \in X$ and open neighbourhoods V_α of x_α $(\alpha \in \kappa)$ with the property that $x_\beta \in V_\alpha$ implies $\beta = \alpha$; while $\widehat{hc}(X)$ is the minimum of all κ for which there are no such discrete sequences of length κ.

A space X is: *hereditarily separable* if $hd(X) \leqslant \aleph_0$, *hereditarily Lindelöf* if $hL(X) \leqslant \aleph_0$, or *hereditarily c.c.c.* if $hc(X) \leqslant \aleph_0$. A space X is an *S-space* if it is hereditarily separable but not hereditarily Lindelöf, and is an *L-space* if it is hereditarily Lindelöf but not hereditarily separable.

The sup = max *problem.* The question of when (for example) $hc < \widehat{hc}$ is an instance of the 'sup = max' problem. Juhasz in [7, Chapter 4] has recorded numerous results about sup = max problems. Another source is [6]. We list the results which we will have use of in the next section. It is clear that the problem has a positive answer when $hc(X)$ is a successor cardinal.

Theorem 1 (Juhasz [7, p. 86]). *Let κ be a strong limit cardinal with uncountable cofinality. Then a space X with $hc(X) = \kappa$ satisfies $hc(X) < \widehat{hc}(X)$.*

Theorem 2 [6, Theorem 8.1]. *Let X be a metrizable space. Then $hc(X) < \widehat{hc}(X)$.*

Open universals. In each of the results below, the space X has an open universal U parametrized by the space Y.

Lemma 3. *If X' is a subspace of X, then $U \cap (X' \times Y)$ is an open universal set for X' parametrized by Y.*

Theorem 4. *The following are equivalent for a regular cardinal κ:*

(1) $2^{\aleph_0} < 2^{\kappa}$.
(2) *If X has an open universal set parametrized by Y, with $|Y| \leqslant \mathfrak{c}$, then $hc(X) < \kappa$.*
(3) *Every space X with an open universal set parametrized by a compact first countable separable space Y has hereditary cellularity less than κ.*

Theorem 5.
(1) $w(X) \leqslant nw(Y)$;
(2) $hd(X) \leqslant hL(Y)$;
(3) $hL(X) \leqslant hd(Y)$;
(4) $hc(X) \leqslant hc(Y)$.

Corollary 6. *The following are equivalent*:
(1) *X has an open universal set parametrized by 2^{ω}.*
(2) *X has an open universal set parametrized by some cosmic space.*
(3) *X is separable and metrizable.*

Theorem 7. *Suppose X has a G_δ-diagonal. Then*:
(1) $hLt(X^{\omega}) \leqslant hd(Y)$; (2) $hd(X^{\omega}) \leqslant hL(Y)$; (3) $hc(X^{\omega}) \leqslant hc(Y)$.
$\widehat{(1)}$ $\widehat{hL}(X^{\omega}) \leqslant \widehat{hd}(Y)$; $\widehat{(2)}$ $\widehat{hd}(X^{\omega}) \leqslant \widehat{hL}(Y)$; $\widehat{(3)}$ $\widehat{hc}(X^{\omega}) \leqslant \widehat{hc}(Y)$.

The 'hat' versions ($\widehat{(1)}$ etc.) are not explicitly stated in [5, Theorem 22], but follow immediately from the argument given for the 'non-hatted' case.

Lemma 8. *If parametrizes X, $w(X) \leqslant hL(X \times Y)$.*

3. Parametrizing discrete spaces

For a cardinal μ, and a family $\mathcal{C}$, define $\langle \mathcal{C} \rangle_\mu = \{\bigcup \mathcal{D}\colon \mathcal{D} \in [\mathcal{C}]^{\leqslant \mu}\}$. Now define the *$\mu$-weight*, denoted w_μ, of a space (X, τ), to be $\min\{|\mathcal{B}|\colon \mathcal{B}$ is a family of open sets such that $\langle \mathcal{B} \rangle_\mu = \tau\}$. The case when μ is $\aleph_0$ is sufficiently important that we write σw for $w_{\aleph_0}$.

We note the following basic results on μ-weight, and follow that with the relationship between μ-weight and open universals.

Lemma 9. *Let (X, τ) be a space; μ_1 and μ_2 be infinite cardinals, with $\mu_1 \geqslant \mu_2$; and $n \in \omega$. Then*:

$$w(X) = w_{hL(X)}(X) \leqslant w_{\mu_1}(X) \leqslant w_{\mu_2}(X) \leqslant w_n(X) = |\tau|.$$

Proposition 10. *The following are equivalent for a space X, and cardinals λ and μ with $\mu \leqslant \lambda$*:
(1) *The space $D(\lambda)^{\mu}$ parametrizes an open universal for X.*
(2) *There is a space Y of weight $\leqslant \lambda$ and character $\leqslant \mu$ parametrizing X.*
(3) *The μ-weight of X is no more than λ.*

Proof. (1) $\Rightarrow$ (2): The space $D(\lambda)^\mu$ has weight λ and character μ.

(2) $\Rightarrow$ (3): From [5, Theorem 10], there is a base for X of size no greater than $nw(Y)$. It is easy to see that this base is a μ-base.

(3) $\Rightarrow$ (1): We can use the same arguments as in [5, Lemma 2]. □

Let κ, λ and μ be cardinals. We will say that $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ holds if and only if there is a $\mathcal{C}$ contained in $\mathcal{P}(\kappa)$ such that $|\mathcal{C}| \leqslant \lambda$ and $\langle\mathcal{C}\rangle_\mu = \mathcal{P}(\kappa)$. The next proposition, relating the combinatorial statement $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ to efficient parametrization of discrete spaces, is simply a restatement of Proposition 10 in this special case.

Proposition 11. *The following are equivalent*:

(1) $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ *holds.*

(2) *There is a space Y, with $w(Y) \leqslant \lambda$ and $\chi(Y) \leqslant \mu$ parametrizing $D(\kappa)$.*

(3) *The space $D(\lambda)^\mu$ parametrizes $D(\kappa)$.*

Note that for all κ, $\mathcal{P}\kappa \equiv \langle\kappa\rangle_\kappa$ is true. If $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ is true and if $\gamma \geqslant \lambda$ and $\nu \geqslant \mu$, then $\mathcal{P}\kappa \equiv \langle\gamma\rangle_\nu$ is true. Restrictions are recorded below; first **ZFC** results, then consistency results. The latter are the central technical data for this paper.

Theorem 12. *Let κ, λ and μ be cardinals.*

(1) $\mathcal{P}\kappa \equiv \langle 2^\kappa\rangle_1$ *is true.*

(2) *If $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ holds, then $\lambda^\mu \geqslant 2^\kappa$.*

Suppose $cf(\kappa) > \mu$:

(3) $\mathcal{P}\kappa \equiv \langle\kappa\rangle_\mu$ *is false.*

(4) *If $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ holds, then $\lambda \geqslant |\mathcal{A}|$ for any maximal almost disjoint family $\mathcal{A}$ of subsets of κ.*

Proof. *Ad* (1): Clear.

Ad (2): Immediate from $|\mathcal{C}| = \lambda$ and $|\langle\mathcal{C}\rangle_\mu| = |\mathcal{P}(\kappa)|$.

Ad (3): Suppose not. Note that we can find a family $\{\Gamma_\alpha\}_{\alpha\in\kappa} \subseteq [\kappa]^\kappa$ of disjoint subsets such that $\kappa = \bigcup_{\alpha\in\kappa}\Gamma_\alpha$. List the elements of $\mathcal{C}$ as $\{C_\alpha\}_{\alpha\in\kappa}$, and, for each $\alpha < \kappa$, let $\mathcal{C}_\alpha = \{C_\beta\}_{\beta\leqslant\alpha}$.

Fix $\alpha < \kappa$. The set $\Lambda_\alpha = \{\beta \in \Gamma_\alpha\colon$ there is $\gamma \leqslant \alpha$ such that $\Gamma_\alpha \cap C_\gamma = \{\beta\}\}$ has size $< \kappa$. So we may pick β_α in $\Gamma_\alpha \setminus \Lambda_\alpha$.

Let $A = \{\beta_\alpha\}_{\alpha<\kappa}$. Then $A \subseteq \kappa$, and, using the cofinality restriction, there is an $\alpha_0 < \kappa$ such that $A \in \langle\mathcal{C}_{\alpha_0}\rangle_\mu$. Now β_{α_0} is in A, so there is $\beta < \alpha_0$ such that $\beta_{\alpha_0} \in C_\beta \subseteq A$. By definition of Λ_{α_0}, there is $x \in C_\beta \cap \Gamma_{\alpha_0} \setminus \{\beta_{\alpha_0}\} \subseteq A \cap \Gamma_{\alpha_0} \setminus \{\beta_{\alpha_0}\}$. Then there is $\gamma < \kappa$, $\gamma \neq \alpha_0$, such that $x = \beta_\gamma$. Therefore, $\beta_\gamma \in \Gamma_{\alpha_0} \cap \Gamma_\gamma$, which is a contradiction.

Ad (4): Suppose that $\mathcal{A}$ is a maximal almost disjoint family on κ, and that each element A of $\mathcal{A}$ is a μ-union of elements of $\mathcal{C}$. Since $cf(\kappa) > \mu$, for each A in $\mathcal{A}$, there is a $C(A) \in \mathcal{C}$ such that $|C(A)| = \kappa$ and $C(A) \subseteq A$. This implies that $|\mathcal{C}| \geqslant |\mathcal{A}|$. □

Theorem 13. *Let κ be a cardinal, with $cf(\kappa) > \mu$:*

(1) (**GCH**) *$\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ holds if and only if $\lambda \geqslant 2^\kappa$.*
(2) (**MA** *and $\aleph_1 \leqslant \kappa < 2^{\aleph_0}$*) *$\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ holds if and only if $\lambda \geqslant 2^\omega$* (*and of course $2^{\aleph_0} = 2^{\aleph_1} = 2^\kappa$, so this contrasts with* (1) *above*).
(3) *It is consistent that*:
For all n in ω, we have $\mathcal{P}\aleph_n \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ holding, and $2^{\aleph_n} = \aleph_{\omega+1} = \aleph_\omega^{\aleph_0}$.

Proof. *Ad* (1): Immediate from Theorem 12 (2 and 3).

Ad (2): This follows from the observation that under **MA**, there is an uncountable subset of ω_1 which fails to contain any one of a given collection of less than $2^{\aleph_0}$ uncountable sets. This uses a Cohen real argument, which we proceed to describe.

Suppose (**MA** and $\kappa < 2^{\aleph_0}$) holds, but the claim is false, with $\mathcal{C}$ witnessing $\mathcal{P}\kappa \equiv \langle\lambda\rangle_\mu$ true. Let $\mathbb{P} = \{(E, F) \in [\kappa]^{<\omega} \times [\kappa]^{<\omega}\colon\ E \subseteq F\}$, with order that satisfies the c.c.c. defined via

$$\left(E', F'\right) \leqslant (E, F) \quad \text{iff} \quad F' \supseteq F \text{ and } E' \cap F = E.$$

Let $D_C = \{(E, F) \in \mathbb{P}\colon\ F \cap C \neq E \cap C\}$ for each $C \in \mathcal{C}$ of size κ, and $\Delta_\alpha = \{(E, F) \in \mathbb{P}\colon\ \exists\beta \geqslant \alpha(\beta \in E)\}$ for each $\alpha \in \kappa$. These are dense in $\mathbb{P}$. Let G be a generic filter meeting the given dense sets, and $S = \bigcup\{E\colon\ (E, F) \in G\} \in [\kappa]^\kappa$.

We claim that there is no element of $\mathcal{C}$ of size κ that is also a subset of S. If C is such an element, then there is $(E, F) \in G \cap D_C$. That is to say, $F \cap C \neq E \cap C$, witnessed by some α. Suppose $\alpha \in S$. Then there is $(E', F') \in G$ such that $\alpha \in E'$. Let (E^*, F^*) be in G below both (E, F) and (E', F'). Then $\alpha \in E^*$. But $(E, F) \geqslant (E^*, F^*)$ means that $E^* \cap F = E$ and this would contain α, which contradicts the assumption $\alpha \notin E$.

Ad (3): Let G be $\langle V, Fn(\aleph_\omega, \omega)\rangle$-generic, where **GCH** holds in V.

We say that a name for a subset of an ordinal λ is *nice* if it is of the form $\{\langle p, \check{\alpha}\rangle\colon\ \alpha \in \lambda,\ p \in \mathcal{A}_\alpha\}$, where $\check{\alpha}$ is a standard name for α, and $\mathcal{A}_\alpha$ is some antichain.

We say that a nice name $\dot{T}$ for a subset of λ is λ-*bounded* if whenever $\langle p, \check{\alpha}\rangle \in \dot{T}$, $dom(p) \subseteq \lambda$. Note that there are at most $\aleph_\delta$ pairs $\langle p, \check{\alpha}\rangle$ such that $\alpha < \aleph_\delta$ and $dom(p) \subseteq \aleph_\delta$; so there are only at most $2^{\aleph_\delta} = \aleph_{\delta+1}$ $\aleph_\delta$-bounded nice names.

Next we observe that every nice name $\dot{S}$ for a subset of $\aleph_\omega$ is the union of names $\dot{S}_n$, for $n \in \omega$, where $\dot{S}_n$ is $\aleph_n$-bounded: simply define $\dot{S}_n$ to be $\{\langle p, \check{\alpha}\rangle \in \dot{S}\colon\ dom(p) \subseteq \aleph_n,\ \alpha < \aleph_n\}$. And, since $\dot{S} = \bigcup_{n\in\omega} \dot{S}_n$, so also $\Vdash \dot{S} = \bigcup_{n\in\omega} \dot{S}_n$.

In other words, in the generic extension $V[G]$, every subset of $\aleph_\omega$ is a union of countably many sets, each of which has, for some n, an $\aleph_n$-bounded nice name. Let $\mathcal{G}$ be the set of all sets with, for some n, an $\aleph_n$-bounded nice name. Then what we have just established is that $\mathcal{P}\aleph_\omega = \langle\mathcal{G}\rangle_{\aleph_0}$. But the cardinality of $\mathcal{G}$ is at most $\sum_{n\in\omega} \aleph_{n+1} = \aleph_\omega$. So $\mathcal{P}\aleph_\omega \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ holds, so a fortiori $\mathcal{P}\aleph_n \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ holds.

In the generic extension, $2^{\aleph_n} = \aleph_{\omega+1} = {\aleph_\omega}^{\aleph_0}$. So, of course, $\aleph_\omega$ is not a strong limit. $\square$

Note that if Y parametrizes an open universal of X, then it parametrizes open universals of any subspace of X. The above results then yield bounds on $\widehat{hc}$ of a space X with open universal parametrized by a space Y in terms of the weight and character of Y. Where the 'sup = max' has a positive answer for hc (see Theorems 1 and 2), the bound is *strict* on the hereditary cellularity. For simplicity we only give the result for σw.

Theorem 14. *Let a first countable space Y parametrize an open universal for a space X. Let κ be a cardinal with uncountable cofinality. Then $\widehat{hc}(X) \leqslant \kappa$ if*:

(i) $w(Y) \leqslant \kappa$ *or*

(ii) $w(Y) < 2^\kappa$ *and* $(w(Y)^{\aleph_0} < 2^\kappa)$ *or* $(\kappa^+ = 2^\kappa)$ *or* (**MA** *and* $\kappa < 2^{\aleph_0}$).

And hence, $hc(X) < \kappa$ *if, in addition*:

(a) κ *is a successor, or*

(b) κ *is a strong limit, or*

(c) X *is metrizable.*

Suppose a space X has $\sigma w(X) \leqslant \aleph_1$ (equivalently, has an open universal parametrized by a first countable space of weight $\aleph_1$). Then we deduce from Theorem 14 that X is hereditarily c.c.c. and weight $\leqslant \aleph_1$. Further, we know if a space X has weight $\leqslant \aleph_1$ and is hereditarily Lindelöf, then X has $\sigma w \leqslant \aleph_1$. It is natural to ask:

1. Does $\sigma w(X) \leqslant \aleph_1$ imply X is hereditarily Lindelöf? Hereditarily separable?
2. Does $w(X) \leqslant \aleph_1$ and X hereditarily c.c.c. imply $\sigma w(X) \leqslant \aleph_1$?

Both (1) and (2) are true when the (consistent) statement (S) *there are no S-spaces* holds.

Proposition 15. (S) *The following are equivalent for a space X*:

(a) $\sigma w(X) \leqslant \aleph_1$.

(b) $w(X) \leqslant \aleph_1$ *and* $hc(X) \leqslant \aleph_0$.

(c) $w(X) \leqslant \aleph_1$ *and* $hL(X) \leqslant \aleph_0$.

But both (1) and (2) can be consistently false.

Example 16.

(1a) (**CH**) Let X be the Kunen line. Then X is not hereditarily Lindelöf but $\sigma w(X) = \aleph_1$.

(1b) Any L-space that has weight $\aleph_1$ is not hereditarily separable but has σ-weight $\aleph_1$. Assuming $\diamondsuit$, let X be a Souslin line. Then X is not hereditarily separable but $\sigma w(X) = \aleph_1$.

(2) It is consistent that there is a subspace X of 2^{ω_1} so that $|X| = 2^{\aleph_1} > \aleph_2$ and X is hereditarily separable. Then $w(X) = \aleph_1$, X is hereditarily c.c.c. but $\sigma w(X) > \aleph_1$.

Proof. *Ad* (1a): For properties of the Kunen line, see [10, Section 4]. We note that it is a locally countable S-space, such that each Kunen-line open set is the union of an Euclidean open set with countably many countable Kunen-line open sets. Thus a σ-base for the Kunen line is a countable base of the Euclidean topology, together with a countable local base of Kunen-line open sets for each of the $\aleph_1$ many points of the Kunen line.

Ad (1b): Any base of a hereditarily Lindelöf space is a σ-base.

Ad (2): The space is the HFD described in [10, Section 5.1]. This is a hereditarily separable subspace of 2^{ω_1} of size $2^{\aleph_1}$, existing in a model of set theory where $2^{\aleph_1}$ is strictly greater than $\aleph_2$, and the Continuum Hypothesis holds. Further, X has exactly $2^{\aleph_1}$ many open sets. It is then clear that there can be no σ-base for X of size $\aleph_1$, since **CH** holds. $\square$

4. First countable parametrization of $\mathcal{W}(\kappa)$

Our first result gives a strong restriction on the cardinals admitting first countable parametrizations of $\mathcal{W}(\kappa)$.

Proposition 17. *If a first countable space Z, with a G_δ-diagonal, has a self open universal, then the cofinality of $w(Z)$ is countable.*

Proof. Let $w(Z) = \kappa$. Let U be an open universal for Z parametrized by Z. Suppose, for a contradiction, that $cf(\kappa) > \omega$.

Since Z is first countable it follows from Theorem 14 and the cofinality restriction, that Z contains no discrete subspaces of size κ. Hence, as Z has a G_δ-diagonal, by Theorem $7\widehat{(3)}$, Z^2 contains no discrete subspaces of size κ. So Z has either no left separated subspaces of size κ, or no right separated subspaces of size κ. In the latter case, by Theortem $7\widehat{(2)}$, it must also have no left separated subspaces of size κ. Applying Theorem $7\widehat{(1)}$ again, we now see that Z^2 contains no right separated subspaces of size κ. Take a subcover $\mathcal{V}$ of $\{V \times W\colon\ V, W$ are open and $V \times W \subseteq U\}$, of minimal size. Then, as Z^2 contains no right separated subspaces of size κ (or larger), $|\mathcal{V}| < \kappa$. But this implies that $\mathcal{B} = \{V\colon\ V \times W \in \mathcal{V}\}$ is a base for Z of size $< \kappa$, contradicting $w(Z) = \kappa$.

In a similar way, we obtain $w(Z) = \kappa$ if Z does not admit a left separated subspace of size κ. $\square$

Recall that for a cardinal κ, which is the limit of a sequence $(\kappa_n)_{n\in\omega}$ of cardinal numbers, we define $C_\kappa = \prod_{n\in\omega} D(\kappa_n)$. Note that $C_{\aleph_0}$ is the Cantor set. For larger κ, C_κ is a metrizable space sharing many of the properties of the Cantor set and the generalised Cantor cubes. Critically, the C_κ's (like the Cantor cubes) efficiently parametrize all spaces of the same or lesser weight, provided κ is a strong limit.

Theorem 18. *For each strong limit cardinal, κ, of countable cofinality, the metrizable space C_κ of weight κ, parametrizes $\mathcal{W}(\kappa)$.*

Proof. Fix κ_n converging up to κ with $\kappa_{n+1} \geqslant 2^{\kappa_n}$ for all $n \in \omega$; and fix surjections ϕ_{n+1} of κ_{n+1} onto 2^{κ_n}. Define $Y = \prod_{n\geqslant 1} D(\kappa_n)$. Then Y is homeomorphic to C_κ. Let $\mathcal{B}$ be a base for a space X in $\mathcal{W}(\kappa)$, where $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{B}_n$ and $\mathcal{B}_n = \{B^n_\alpha\}_{\alpha\in\kappa_n}$.

For a fixed y in Y and $n \geqslant 1$, noting that $\phi_n(y(n))$ is a function from κ_{n-1} to $\{0, 1\}$, define $U(y, n)$ to be $\bigcup\{B^{n-1}_\alpha\colon\ \phi_n(y(n))(\alpha) = 1\}$. Now define $U = \bigcup_{y\in Y} U^y \times \{y\}$ where $U^y = \bigcup_{n\geqslant 1} U(y, n)$.

One can check that U is open. Since, for an open subset V of X, $V = \bigcup_{n\in\omega} V_n$ where $V_n = \bigcup\{B^n_\alpha\colon\ B^n_\alpha \subseteq V\}$, it follows that U is an open universal for X parametrized by Y. $\square$

Parts (1)–(3) of Theorem A in the introduction, follow from Proposition 17 and Theorem 18, observing that, under **GCH**, cardinals of countable cofinality are all strong limits. Parts (4) and (5) of Theorem are proved next.

Proposition 19. *It is consistent that*:

No first countable space of weight $\aleph_\omega$ *parametrizes* $D(\aleph_\omega)$.

Proof. We give two models of Set Theory where the claim holds.

The first is a model of **MA** and $2^{\aleph_0} = \aleph_{\omega+1}$. In this case, $\mathcal{P}\aleph_1 \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ is false by Theorem 13(2). Hence Proposition 11 gives the claim.

Alternatively, forcing with $Fn(\aleph_{\omega+2}, 2, \aleph_1)$ over a ground model satisfying **GCH** yields a model in which ${\aleph_\omega}^{\aleph_0} = \aleph_{\omega+1}$ and $2^{\aleph_\omega} \geqslant \aleph_{\omega+2}$. This is because the forcing (being countably closed) creates no new countable sets of ordinals, so ${\aleph_\omega}^{\aleph_0}$ is unaltered; while $2^{\aleph_1}$ becomes $\aleph_{\omega+2}$, and so $2^{\aleph_\omega} \geqslant \aleph_{\omega+2}$. The cardinal arithmetic then tells us that $\mathcal{P}\aleph_\omega \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ is false, by Theorem 12(2); and again the claim follows. $\square$

Theorem 20. *It is consistent that* $C_{\aleph_\omega}$ *parametrizes* $\mathcal{W}(\aleph_\omega)$ *but* $\aleph_\omega$ *is not a strong limit.*

Proof. Working in the model of Theorem 13(3), since $\mathcal{P}\aleph_n \equiv \langle\aleph_\omega\rangle_{\aleph_0}$ holds for all n, it is straightforward to modify the proof of Theorem 18 and show that $C_{\aleph_\omega}$ does indeed parametrize $\mathcal{W}(\aleph_\omega)$. Since $2^{\aleph_0} > \aleph_\omega$ in our model, $\aleph_\omega$ is not a strong limit. $\square$

We now compare the properties of the standard Cantor set, $C_{\aleph_0}$, with that of its larger analogues, C_κ. Each uncountable Polish space contains a Cantor subspace [8, Section 6]. The proof can be modified to show the following results.

Proposition 21. *Let* κ *be a cardinal of countable cofinality, and* X *be a completely metrizable space of weight* κ. *If one of* (1) *or* (2) *holds, then* X *contains* C_κ *as a subspace.*

(1) *Every open set of* X *has size at least* κ.

(2) *The space* X *has cardinality greater than* κ.

From now on until the end of this section, assume κ is fixed to be a strong limit cardinal with countable cofinality, say κ is the limit of $(\kappa_n)_{n\in\omega}$, and there are fixed surjections ϕ_{n+1} of κ_{n+1} onto 2^{κ_n}.

Definition 22. A set A is a κ-metric Bernstein set if it is a subset of C_κ not containing a homeomorph of C_κ but meeting all subspaces of C_κ homeomorphic to C_κ.

Lemma 23. *There exist* κ*-metric Bernstein sets.*

Proof. Firstly, because the space C_κ is completely metrizable, each completely metrizable subspace of a metrizable space is a G_δ [8, Theorem 3.11], and because there are 2^κ many G_δ-subsets of C_κ, there are at most 2^κ many subspaces of C_κ homeomorphic to C_κ.

Secondly, we note that from Theorem 18, $2^\kappa \leqslant \kappa^\omega = |C_\kappa|$ is true.

Following the construction of the Bernstein sets of the real line and using the two facts above will give a κ-metric Bernstein set. $\square$

Theorem 24. *Every* κ*-metric Bernstein set parametrizes* $\mathcal{W}(\kappa)$.

Proof. Let A be a κ-metric Bernstein set (in C_κ). Let X be a space in $\mathcal{W}(\kappa)$, with basis $\mathcal{B}$, where $\mathcal{B}$ is the increasing union of $\mathcal{B}_n$, $\mathcal{B}_n = \{B^n_\alpha\}_{\alpha \in \kappa_n}$, and, in this enumeration, each element of $\mathcal{B}_n$ is repeated κ_n times.

Split C_κ into two clopen pieces both homeomorphic to C_κ. This splits A into two non-empty clopen parts, A_0 and A_1 say, where A_1 is a κ-metric Bernstein subset of the second copy of C_κ. We may use A_0 to parametrize the empty set (many times over!), and so, without loss of generality, restrict ourselves to showing that we can parametrize all non-empty open subsets of X by A a κ-metric Bernstein subset of $Y = \prod_{n \geqslant 1} D(\kappa_n)$.

Let U_0 be the open universal for X parametrized by Y as was defined in the proof of Theorem 18; and let $U = (X \times A) \cap U_0$. We are going to prove that U is an open universal for X parametrized by A. To do this, it is sufficient to show that, for each non-empty open subset V of X, the set $V_Y = \{y \in Y\colon\ U_0^y = V\}$ meets A. By the properties of κ-metric Bernstein sets, there is a non-empty intersection, provided V_Y contains a copy of C_κ.

Fix V as above. Fix y_0 in V_Y. As V is non-empty there is an $N \geqslant 1$ and α_N in κ_{N-1} so that $\phi_N(y_0(N))(\alpha_N) = 1$. As the $\mathcal{B}_n$'s are increasing, for each $m \geqslant N$ we can pick α_m in κ_{m-1} so that $\phi_m(y_0(m))(\alpha_m) = 1$. For $m \geqslant N$, let $\Lambda_m = \{\alpha \in \kappa_{m-1}\colon\ B^{m-1}_\alpha = B^{m-1}_{\alpha_m}\}$. Since elements of $\mathcal{B}_{m-1}$ are repeated κ_{m-1} times, Λ_m has cardinality κ_{m-1}, for all $m \geqslant N$. Hence, $T = \{y \in Y\colon\ y(k) = y_0(k)$ for $k < N$ and $\phi_m(y(m))(\alpha_m) = 1$, $\phi_m(y(m))(\alpha) = \phi_m(y_0(m))(\alpha)$ $(\alpha \in \kappa_{m-1} \setminus \Lambda_m)$ for $m \geqslant N\}$, is a homeomorph of C_κ contained in V_Y, as desired. □

5. Self parametrizing spaces

A second (the first is Proposition 17) necessary condition for the existence of open self-universal sets is as follows.

Proposition 25. *Let Z be a space with a G_δ-diagonal and an open universal set parametrized by itself. Then $w(Z) \leqslant hc(Z)$. In particular, if Z is hereditarily c.c.c., then it is metrizable.*

Proof. We know from Theorem 7(3) that $hc(Z^2) = hc(Z) = \kappa$. By an easy argument, it follows that either $hd(Z) \leqslant \kappa$ or $hL(Z) \leqslant \kappa$. If $hL(Z) \leqslant \kappa$, then since Z has an open universal set parametrized by itself, $hd(Z) \leqslant \kappa$, by Theorem 5. Therefore by Theorem 7(1) and Lemma 8, we have $hL(Z^2) \leqslant \kappa$, leading to $w(X) \leqslant \kappa$. □

The statement (B) is *every Boolean algebra with no uncountable weak antichains is countable*. In [2] it is pointed out that techniques of Shelah show that (B) and $2^{\aleph_0} < 2^{\aleph_1}$ is consistent. (See [1,2] for more details on (B).)

Theorem 26 (($2^{\aleph_0} < 2^{\aleph_1}$) and (B)). *Every compact, first countable, zero-dimensional space with a self parametrized open universal, is metrizable.*

Proof. Let Z be as in the statement of the theorem; and let X and Y be two copies of Z. Then by Arhangelskii's theorem, $|Y| \leqslant 2^{\aleph_0}$, and as $2^{\aleph_0} < 2^{\aleph_1}$, X must be hereditarily c.c.c. (Theorem 4). Now Y is a hereditarily c.c.c. space parametrizing an open universal for the compact zero-dimensional X. The claim, that X (and so Z) is metrizable, follows from [5, Theorem 28]. □

Example 27. It is consistent that there is a compact, zero-dimensional and first countable space, of weight $\aleph_\omega$, parametrizing $\mathcal{W}(\aleph_\omega)$.

Proof. We work in the model of Theorem 13(3). In that model $\aleph_\omega < 2^{\aleph_0}$. So we may pick $D = \bigcup_{n\in\omega} D_n$ contained in the Cantor set with $|D_n| = \aleph_n$. Let Y_n be the subspace of the Alexandrov duplicate of the Cantor set, consisting of all the non-isolated points along with the isolated points corresponding to D_n. Note that Y_n has weight $\aleph_n$ and contains a copy of $D(\aleph_n)$. Let Y be the product of the Y_n ($n \in \omega$). Then Y is a compact, zero-dimensional, first countable space, of weight $\aleph_\omega$, containing $C_{\aleph_\omega}$, and thus it parametrizes $\mathcal{W}(\aleph_\omega)$. □

Compactness is essential in Example 27, as the next example demonstrates. Recall that $\beth_1 = 2^{\aleph_0}$, inductively $\beth_{n+1} = 2^{\beth_n}$, and $\beth_\omega = \lim_{n\in\omega} \beth_n$. Then $\beth_\omega$ is a strong limit cardinal of countable cofinality.

Example 28. There is a first countable space with G_δ-diagonal parametrizing itself which is not metrizable.

Proof. Let S be the Sorgenfrey line. Let Y be the sum of S and $C_{\beth_\omega}$. Then Y is first countable, has a G_δ-diagonal, and, since it contains $C_{\beth_\omega}$, parametrizes $\mathcal{W}(\beth_\omega)$, and, in particular, parametrizes itself. □

6. Parametrizing $\mathcal{W}(\kappa)$ weight efficiently

Note that if a space of weight κ contains a copy of the Cantor cube 2^κ, then it parametrizes $\mathcal{W}(\kappa)$. But this property does not determine those spaces which parametrize in a weight efficient manner a $\mathcal{W}(\kappa)$.

Recall that a Bernstein subset of the Cantor set, C, is one which meets all copies of the Cantor set in C, but which contains none. For an arbitrary infinite cardinal κ we similarly define a *κ-Cantor Bernstein set* to be a subspace of 2^κ meeting all subspaces of 2^κ homeomorphic to 2^κ but containing no such subspaces. The standard argument for the Cantor set proves equally well that κ-Cantor Bernstein sets exist for all infinite κ.

The proof of the following result is similar to, but somewhat simpler than, that of Theorem 24.

Theorem 29. *Fix an infinite cardinal κ. Let B be any κ-Cantor Bernstein set. Then B parametrizes $\mathcal{W}(\kappa)$, but does not contain a copy of 2^κ.*

Proof. Let X be in $\mathcal{W}(\kappa)$. Split 2^κ into two clopen pieces both homeomorphic to 2^κ. This splits B into two non-empty clopen parts, B_0 and B_1 say, where B_1 is a κ-Cantor Bernstein subset of the second copy of 2^κ. We may use B_0 to parametrize the empty set, and so, without loss of generality, restrict ourselves to showing that we can parametrize all non-empty open subsets of X by B. Let U_0 be the open universal for X parametrized by 2^κ corresponding to some basis $\mathcal{B} = \{B_\alpha\}_{\alpha\in\kappa}$ for X as given by [5, Lemma 2]; and let $U = (X \times B) \cap U_0$. To complete the proof it is sufficient to show that, for a non-empty open subset, V say, of X, the set $V_{2^\kappa} = \{f \in 2^\kappa\colon U_0^f = V\}$ meets B. By the properties of κ-Cantor Bernstein sets there is a non-empty intersection provided V_{2^κ} contains a copy of 2^κ.

Fix V as above. Fix f_0 in V_{2^κ}. As V is non-empty there is an α_0 so that $f_0(\alpha_0) = 1$. Let $\Lambda = \{\alpha\colon B_\alpha \subset B_{\alpha_0}\}$. Then $|\Lambda| = \kappa$ and any f in 2^κ which coincides with f_0 off Λ is in V_{2^κ}. Clearly these f form a subspace homeomorphic to 2^κ. $\square$

Observe that any infinite space contains a homeomorph of $D(\aleph_0)$. Hence if a space Y does not parametrize $D(\aleph_0)$, then Y does not parametrize any infinite space. Note also that if a space Y has size strictly less than $\mathfrak{c}$, then (for the most basic of reasons) it can not parametrize $D(\aleph_0)$.

Example 30. (**MA**) There is a subspace of the Cantor set, of size $\mathfrak{c}$, which does not parametrize $D(\aleph_0)$.

Proof. For concreteness, consider $D(\aleph_0)$ to have underlying set ω. Let $\{U_\alpha\}_{\alpha\in\mathfrak{c}}$ be the collection of all open subsets of the product of ω and the Cantor set. We diagonalise through this collection of potential open universals.

Define $\psi_\alpha : 2^\omega \to \mathcal{P}(\omega)$ via $\psi_\alpha(y) = U_\alpha^y$. Endow the Cantor set with the product measure, m. Note that the measure algebra of m satisfies the c.c.c.

Suppose we have picked $y_\beta \in 2^\omega$ and $S_\beta \in \mathcal{P}(\omega)$ for all $\beta < \alpha < 2^\omega$, such that, for each $\beta < \alpha$, $y_\beta \notin \{y_\gamma\}_{\gamma<\beta} \cup \bigcup_{\gamma\leqslant\beta} \psi_\gamma^{-1}(S_\gamma)$ and $m(\psi_\beta^{-1}(S_\beta)) = 0$. Let $E_\alpha = \{\psi_\alpha^{-1}(S)\colon S \in \mathcal{P}(\omega) \wedge \forall\beta < \alpha(y_\beta \notin \psi_\alpha^{-1}(S))\}$. Note that this set either contains the empty set or has continuum cardinality.

As ω is countable, each $S \in \mathcal{P}(\omega)$ is countable. Therefore for all $\alpha \in \mathfrak{c}$, $\psi_\alpha^{-1}(S)$ is a G_δ-subset of the Cantor set. Hence $\psi_\alpha^{-1}(S)$ is measurable for each subset S of ω.

Since E_α either contains the empty set or is uncountable, there is an $S_\alpha \in \mathcal{P}(\omega)$ such that $\psi_\alpha^{-1}(S_\alpha) \in E_\alpha$ and $m(\psi_\alpha^{-1}(S_\alpha)) = 0$. Moreover, because we are assuming Martin's Axiom, the set $\{y_\beta\}_{\beta<\alpha} \cup \bigcup_{\beta\leqslant\alpha} \psi_\beta^{-1}(S_\beta)$ has measure zero. So we can choose y_α not in this set.

Let $Y = \{y_\alpha\}_{\alpha\in\mathfrak{c}}$. This subspace of the Cantor set has cardinality $\mathfrak{c}$ and does not parametrize any open universal set of ω. $\square$

On the other hand, such an example described above consistently needs not exist.

Theorem 31. *It is consistent that every metric space of size $\mathfrak{c}$ parametrises $\mathcal{W}(\aleph_0)$.*

Proof. It is known that under the iterated perfect set model, every metric space of size $\mathfrak{c}$ can be continuously mapped onto the continuously mapped onto the closed unit interval (see [11]). The statement is proved when we note that continuous surjective pre-images of superspaces of parametrising spaces for X are also parametrising spaces for X. □

7. Open problems, further developments

There remain a number of open questions concerning parametrizing open universals. In addition, new lines of inquiry suggest themselves.

(1) Is the restriction to spaces with a G_δ-diagonal necessary in Theorem A(2)?

(2) Can the restriction to zero-dimensional spaces be removed from Theorem B(2)?

σ-Generators. Let λ be minimal so that $\mathcal{P}\aleph_1 \equiv \langle\lambda\rangle_{\aleph_0}$ holds. Is it consistent that $\lambda = \aleph_2 < 2^{\aleph_0}$? Is it consistent that $2^{\aleph_0} < 2^{\aleph_1}$ and $\lambda < 2^{\aleph_1}$?

Topology of Cantor cubes. We have introduced a new cardinal invariant, σ-weight, and shown that $2^{\aleph_1}$ can have 'interesting' σ-weight. This suggests further investigation of the topology of Cantor cubes via cardinal invariants.

For example, if $\mathcal{C}$ is a σ-generator for $\mathcal{P}(\aleph_1)$, then identifying subsets of $\aleph_1$ with their characteristic functions, we see that $2^{\aleph_1}$ has a subspace of size $|\mathcal{C}|$ so that every element of $2^{\aleph_1}$ is the limit from below of a sequence from $\mathcal{C}$. Define the *sequential density* of a space X, by $d_s(X) = \min\{|D|\colon$ every element of X is the limit of a sequence on $D\}$. What are the possible values of $d_s(2^{\aleph_1})$? Can it be countable? Can it ever be $\aleph_1$ (cf. Theorem 12.3)?

Function universals. The idea of an universal object is very flexible. For example, let us define a function $F : X \times Y \to \mathbb{R}$ to be *continuous function universal* if F is continuous and for every continuous $f : X \to \mathbb{R}$ there is a $y \in Y$ so that $F(\cdot, y) = f$.

Which (Tychonoff) spaces, X, have a continuous function universal parametrized by a metrizable space of weight $w(X)$? For which cardinals κ does every Tychonoff space of weight κ have a continuous function universal parametrized by a metrizable space of weight κ?

Note that in the above questions, we can take $Y = C(X)$, and F as the evaluation map. Thus we are looking for metrizable 'admissible' topologies on $C(X)$ of minimal weight.

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